J Glob Optim (2009) 43:97–109
DOI 10.1007/s10898-008-9299-5

On G-invex multiobjective programming. Part I. Optimality

Tadeusz Antczak

Received: 10 May 2007 / Accepted: 8 March 2008 / Published online: 12 April 2008

Abstract In this paper, a generalization of convexity, namely G-invexity, is considered in the case of nonlinear multiobjective programming problems where the functions constituting vector optimization problems are differentiable. The modified Karush-Kuhn-Tucker necessary optimality conditions for a certain class of multiobjective programming problems are established. To prove this result, the Kuhn-Tucker constraint qualification and the definition of the Bouligand tangent cone for a set are used. The assumptions on (weak) Pareto optimal solutions are relaxed by means of vector-valued G-invex functions.

Keywords Multiobjective programming · (weak) Pareto optimal solution · (strictly) G-invex vector function with respect to η · G-Karush-Kuhn-Tucker necessary optimality conditions · Kuhn-Tucker constraint qualification

1 Introduction

In the recent years, the analysis of optimization problems with several objectives conflicting with one another has been a focal issue. Such multiobjective optimization problems are useful mathematical models for the investigation of real-world problems, for example, in engineering, economics, and human decision making. An optimal solution to such an optimization problem is ordinarily chosen from the set of all (weak) Pareto optimal solutions to it. Many authors have developed the necessary and/or sufficient optimality conditions for (weak) Pareto optimality in vector optimization problems (see, for example, [10,15–17,20,22,24], and others). On the other hand, the duality theory has been another focal issue for a long time, especially in convex programming.

But in most of such studies an assumption of convexity on the problems was made to prove the sufficiency of optimality conditions (see, for example, [10,15,22,24]). Recently, considerable progress has been made to weaken the convexity hypothesis and so to increase

T. Antczak (✉)
Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, 90-238 Łódź, Poland
e-mail: antczak@math.uni.lodz.pl

the class of optimization problems for which the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. Therefore, several new concepts concerning a generalized convex function have been proposed. There is an important contribution in this direction given by Hanson in [12]. Hanson considered a differentiable function $f: X \to R$, $X \subset R^n$, for which there exists an n-dimensional vector function $\eta: X \times X \to R^n$ such that, for all $x, u \in X$, the inequality

$$f(x) - f(u) \geq \nabla f(u)\eta(x, u) \tag{1}$$

holds. In [8], Craven called functions satisfying (1) invex. Of course, differentiable convex functions are invex with respect to the function $\eta(x, u) = x - u$. After the works of Hanson [12] and Craven [8], other classes of differentiable nonconvex functions have appeared with the intent of generalizing the class of invex functions from different points of view. Ben Israel and Mond [7], Craven and Glover [9], Hanson and Mond [13], Martin [19], Antczak [1] and many others have studied some properties, applications and further generalizations of invex functions. One of such generalizations of invex functions is also G-invexity introduced by Antczak [4] for scalar differentiable functions. He introduced new necessary optimality conditions for differentiable mathematical programming problem. Antczak also applied the introduced G-invexity notion to develop sufficient optimality conditions and new duality results for differentiable mathematical programming problems.

Furthermore, in the natural way, Hanson's definition of invex functions was also extended to the case of differentiable vector-valued functions. Therefore, in the recent years, there have been very popular applications of invexity in multiobjective optimization problems. Craven and Glover [9] characterized the cone-invexity property, for differentiable functions, in terms of Lagrange multipliers. They also established Kuhn-Tucker type optimality conditions and duality theorems for cone invex programs. Jeyakumar and Mond [14] introduced the class of the so- called V-invex functions to prove some optimality and duality results for a larger class of differentiable vector optimization problems than under invexity assumption. Giorgi and Guerraggio [11] introduced some broad classes of generalized invex vector functions for both in the differentiable and nonsmooth case. Further, they used these notions of generalized invexity to extend some results of weak efficiency, efficiency and duality. The results established by Osuna-Gómez et al. [20] are a generalization for the vectorial case of the results obtained by Martin [19] in the scalar case. These results characterized invex functions as those for which their stationary points are global minima. Batista dos Santos et al. [5] also extended optimality results previously established by Martin [19] in the scalar case to the vectorial case. Using the (p, r)-invexity as a generalization of invexity in the vectorial case, Antczak [2] established some optimality and duality results for a larger class of smooth multiobjective programming problems than invex vector optimization problems. The sufficient conditions given in [2] improve and extend the results of Singh [22]. The results established by Jeyakumar and Mond [14] were extended by Antczak [3] to the class of differentiable multiobjective programming problems involving V-r-invex functions.

This paper represents the first part of study concerning multiobjective programming with a new class of vector nonconvex functions. In this work, we study the optimality for a so-called G-multiobjective programming problems. To do this, we define in the paper a new class of differentiable nonconvex vector valued functions, namely vector G-invex (G-incave) functions with respect to η. This class of functions is a generalization of G-invex (G-incave) functions introduced by Antczak [4] for differentiable mathematical programming problems to the vectorial case. The main purpose of this article is to apply vector G-invexity to develop optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints. Considering the concept of a (weak) Pareto solution, we establish the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable vector optimization problems under the Kuhn-Tucker constraint qualification. Moreover, to prove these necessary optimality conditions for differentiable multiobjective programming problems, we also use the definition of the Bouligand tangent cone for a set (in other words, the set of convergence vectors for a set in terminology given by Lin [17]). The G-Karush-Kuhn-Tucker necessary optimality conditions are weaker than the standard Karush-Kuhn-Tucker necessary optimality conditions well-known in the literature. Furthermore, based on the introduced G-Karush-Kuhn-Tucker necessary optimality conditions, we give sufficient optimality for both weak Pareto and Pareto optimality in multiobjective programming problems involving G-invex and G-incave functions with respect to the same function η and with respect to, not necessarily, the same function G. In particular, the sufficient optimality conditions are more useful for some class of vector optimization problems than the sufficient optimality conditions with vector-valued invex functions. The optimality results established in the paper are illustrated by a suitable example of a vector optimization problem involving G-invex (G-incave) functions with respect to the same function η and with respect to, not necessarily, the same function G.

2 Vector *G*-invex functions

In this section, we provide some definitions and some results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper.

For any $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, \ldots, n$;
(iii) $x \leqq y$ if and only if $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$;
(iv) $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.

Throughout the paper, we will use the same notation for row and column vectors when the interpretation is obvious.

We say that a vector $z \in R^n$ is negative if $z \leqq 0$ and strictly negative if $z < 0$.

Definition 1 A function $f: R \to R$ is said to be strictly increasing if and only if

$$\forall x, y \in R \quad x < y \quad \Longrightarrow \quad f(x) < f(y).$$

Now, in the natural way, we generalize the definition of a real-valued G-invex function introduced by Antczak [4] to the vectorial case.

Let $f = (f_1, \ldots, f_k): X \to R^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subset R^n$, and $I_{f_i}(X)$, $i = 1, \ldots, k$, be the range of f_i, that is, the image of X under f_i.

Definition 2 Let $f: X \to R^k$ be a vector-valued differentiable function defined on a nonempty set $X \subset R^n$ and $u \in X$. If there exist a differentiable vector-valued function $G_f = \left(G_{f_1}, \ldots, G_{f_k}\right): R \to R^k$ such that any its component $G_{f_i}: I_{f_i}(X) \to R$ is a strictly increasing function on its domain and a vector-valued function $\eta: X \times X \to R^n$ such that, for all $x \in X$ $(x \neq u)$ and for any $i = 1, \ldots, k$,

$$G_{f_i}\left(f_i(x)\right) - G_{f_i}\left(f_i(u)\right) - G'_{f_i}\left(f_i(u)\right) \nabla f_i(u) \eta(x, u) \geqq 0 \quad (>), \tag{2}$$

then f is said to be a (strictly) vector G_f-invex function at u on X (with respect to η) (or shortly, G-invex function at u on X). If (2) is satisfied for each $u \in X$, then f is vector G_f-invex on X with respect to η.

If a function $f_i, i \in I$, satisfies (2), we will also say that f_i is G_{f_i}-invex function at u on X with respect to η.

Remark 3 In order to define an analogous class of (strictly) vector G_f-incave functions with respect to η, the direction of the inequality in the definition of these functions should be changed to the opposite one.

Remark 4 In the case when $G_{f_i}(a) \equiv a, i \in I$, for any $a \in I_{f_i}(X)$, we obtain a definition of a vector-valued invex function.

Remark 5 In the case when $k = 1$, we obtain a definition of a scalar G-invex function introduced by Antczak [4].

Proposition 6 *Let X be a nonempty open subset of R^n and the differentiable function $f : X \to R^k$ be surjective and let $\nabla f(u)$ be onto for every $u \in X$, and $G_{f_i} : I_{f_i}(X) \to R, i = 1, \ldots, k$, be differentiable real-valued strictly increasing convex functions on their domains. Then, there exists the vector-valued function η: $X \times X \to R^n$ such that the function f is G_f-invex with respect to η.*

Proof To show this, let $x, u \in X$, $w = f_i(x)$, $v = f_i(u)$. Then, by convexity of G_{f_i}, $i = 1, \ldots, k$, we get

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) = G_{f_i}(w) - G_{f_i}(v) \geqq G'_{f_i}(v)(w - v) \tag{3}$$

Since $\nabla f(u)$ is onto, then

$$w - v = \nabla f_i(u)\eta(x, u)$$

is solvable for some $\eta(x, u) \in R^n$, where η: $X \times X \to R^n$. Hence, by (3), we obtain for $i = 1, \ldots, k$,

$$G_{f_i}(f(x)) - G_{f_i}(f(u)) \geqq G'_{f_i}(f_i(u)) \nabla f_i(u)\eta(x, u).$$

This means that f is G_f-invex on X with respect to η and $G_f = \left(G_{f_1}, \ldots, G_{f_k}\right)$. □

3 Optimality conditions in multiobjective programming

In general, a multiobjective programming problem is formulated as the following vector minimization problem:

$$\begin{aligned} \min f(x) = (f_1(x), f_2(x), \ldots, f_k(x)) \\ x \in D, \end{aligned} \tag{VP}$$

where D is a nonempty set of R^n, and f_i denotes a real-valued differentiable function on D.

Before studying optimality in multiobjective programming, one has to define clearly the concepts of optimality and solutions in multiobjective programming problem. Note that, in vector optimization problems there is a multitude of competing definitions and approaches. The dominat ones are now various scalarizations and (weak) Pareto optimality [21]. The (weak) Pareto optimality in multiobjective programming associates the concept of a solution with some property that seems intuitively natural.

Definition 7 A feasible point $\overline{x}$ is said to be a Pareto solution (an efficient solution) for a multiobjective programming problem if and only if there exists no $x \in D$ such that

$$f(x) \leq f(\overline{x}).$$

Definition 8 A feasible point $\overline{x}$ is said to be a weak Pareto solution (a weakly efficient solution, a weak minimum) for a multiobjective programming problem if and only if there exists no $x \in D$ such that

$$f(x) < f(\overline{x}).$$

As follows from the definition of (weak) Pareto optimality, $\overline{x}$ is nonimprovable with respect to the vector cost function f. The quality of nonimprovability provides a complete solution if $\overline{x}$ is unique. However, usually this is not the case, and then one has to find the entire exact set of all Pareto optimality solutions in a multiobjective programming problem.

As it is known [7], a characteristic property of a scalar invex function with respect to η is the fact that each its stationary point is also its global minimum. It turns out that this property can be generalized to the class of vector G-invex functions with respect to η. For this purpose, we have to define adequately the critical point concept for vector-valued functions.

Definition 9 A point $u \in D$ is said to be a vector critical point of a vector function $f: D \rightarrow R^k$, $D \subset R^n$, if there exists a vector $\lambda \in R^k$ with $\lambda \geq 0$ such that $\lambda^T \nabla f(u) = 0$.

In [8], Craven proved that every weakly efficient point is also a vector critical point, that is, the following theorem is true:

Theorem 10 *Let $\overline{x}$ be a weakly efficient solution for (VP). Then, there exists a vector $\overline{\lambda} \in R^k$ with $\overline{\lambda} \geq 0$ such that $\overline{\lambda}^T \nabla f(\overline{x}) = 0$.*

Now, we prove the converse of Theorem 10 using vectorial G-invexity property.

Theorem 11 *Let $\overline{x}$ be a vector critical point for (VP), and let f be a vector G-invex function at $\overline{x}$ with respect to η. Then $\overline{x}$ is a weak Pareto solution for (VP).*

Proof Let $\overline{x}$ be a vector critical point; i.e. there exists $\lambda \in R^k$ with $\lambda \geq 0$ such that $\lambda^T \nabla f(\overline{x}) = 0$. We proceed by contradiction. We suppose that $\overline{x}$ is not a weak Pareto solution for (VP). Then, there exists another point $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) < f(\overline{x}). \tag{4}$$

By assumption, f is a vector G-invex function at $\overline{x}$ with respect to η. Then, by Definition 2, there exists a vector-valued function $G_f := \left(G_{f_1}, \ldots, G_{f_k}\right): R \rightarrow R^k$ and a vector-valued function $\eta: D \times D \rightarrow R^n$ such that (2) is satisfied. As follows from Definition 2, each function $G_{f_i}: I_{f_i}(D) \rightarrow R$, $i = 1, \ldots, k$, is a strictly increasing function on $I_{f_i}(D)$. This implies together with (4), for $i = 1, \ldots, k$,

$$G_{f_i}(f_i(\widetilde{x})) < G_{f_i}(f_i(\overline{x})). \tag{5}$$

Thus, by vector G_f-invexity of f, we have for $i = 1, \ldots, k$,

$$G'_{f_i}(f_i(\overline{x})) \nabla f_i(\overline{x}) \eta(\widetilde{x}, \overline{x}) < 0.$$

Therefore,

$$\nabla f_i(\overline{x}) \eta(\widetilde{x}, \overline{x}) < 0.$$

Then, by Gordan's theorem of the alternative [18], the system

$$\lambda^T \nabla f(\overline{x}) = 0,$$
$$\lambda \in R^k,\ \lambda \geq 0,$$

has no solution for λ. □

Hence, for multiobjective problems with G-invex functions, weakly efficient points are those for which (and only those for which) the gradient vectors of the component functions, valued at that point, are linearly dependent.

Note that there exist multiobjective optimization problems for which the condition contained in Theorem 10 is only a necessary optimality condition to obtain an efficient solution.

Now, we give some useful characterization of (weak) Pareto optimal solutions. To do this, we give now the definition of a (weak) minimal element for the given set.

Definition 12 [17] Let W be a given set in R^k ordered by $\leqq$ or by $<$. Specifically, we call the minimal element of W defined by $\leq$ a minimal vector, and that defined by $<$ a weak minimal vector. Formally speaking, a vector $\overline{z} \in W$ is called a minimal vector in W if there exists no vector z in W such that $z \leq \overline{z}$; it is called a weak minimal vector if there exists no vector z in W such that $z < \overline{z}$.

The following equivalence is immediate:

Theorem 13 *Let $\overline{x}$ be a feasible solution in a multiobjective programming problem and let $G_{f_i}, i = 1, \ldots, k$, be a continuous real-valued strictly increasing function defined on $I_{f_i}(D)$. Further, we denote $W = \left\{\left(G_{f_1}(f_1(x)), \ldots, G_{f_k}(f_k(x))\right) : x \in D\right\} \subset R^k$ and $\overline{z} = \left(G_{f_1}(f_1(\overline{x})), \ldots, G_{f_1}(f_1(\overline{x}))\right) \in W$. Then, $\overline{x}$ is a (weak) Pareto solution in the set of all feasible solutions D for a multiobjective programming problem if and only if the corresponding vector $\overline{z}$ is a (weak) minimal vector in the set W.*

Now, we consider the following constrained multiobjective programming problem (CVP):

$$\begin{aligned} V\text{-minimize } f(x) &:= (f_1(x), \ldots, f_k(x)) \\ g(x) &\leqq 0, \\ h(x) &= 0, \\ x &\in X, \end{aligned} \tag{CVP}$$

where $f_i: X \to R, i \in I = \{1, \ldots, k\}, g_j: X \to R, j \in J = \{1, \ldots, m\}, h_t: X \to R, t \in T = \{1, \ldots, p\}$ are differentiable functions on a nonempty open set $X \subset R^n$.

Let $D = \{x \in X: g_j(x) \leqq 0, j \in J, h_t(x) = 0, t \in T\}$ be the set of all feasible solutions for problem (CVP). Further, we denote by $J(\overline{x}) := \left\{j \in J: g_j(\overline{x}) = 0\right\}$ the set of constraint indices active at $\overline{x} \in D$ and $I(\overline{x}) := \{i \in I: \lambda_i > 0\}$ the objectives indices set for which corresponding Lagrange multiplier is not equal to 0.

To establish the necessary optimality conditions for the considered multiobjective programming problem (CVP), we need the definition of the Bouligand tangent cone to the given set $W \subset R^k$.

Definition 14 Let $W \subset R^k$. The Bouligand tangent cone to W at $\overline{z} \in W$ is the set $C(W, \overline{z})$ of all vectors $q \in R^k$ such that there exist a sequence $\{z_l\}$ in W and a sequence $\{\lambda_l\}$ of strictly positive real numbers such that,

$$\lim_{l\to\infty} z_l = \overline{z},\ \lim_{l\to\infty} \lambda_l = 0,\ \lim_{l\to\infty} \frac{z_l - \overline{z}}{\lambda_l} = q.$$

Remark 15 Note that Lin [17] named any Bouligand tangent vector, that is, any vector $q \in C(W, \overline{z})$, a convergence vector for the set W at $\overline{z}$.

Now, we give the result established by Lin (see Theorem 5.1 [17]).

Theorem 16 *If $\overline{z} \in W$ is locally (weak) minimal vector in the set $W \subset R^k$ then no Bouligand tangent vector for W at $\overline{z}$ is strictly negative.*

In [4], Antczak introduced the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable mathematical programming problem. To prove an analogous result in the vectorial case, we need the Kuhn-Tucker constraint qualification (see, for example, [6] and [23] for the present formulation).

Definition 17 Let D be a set of all feasible solutions in the multiobjective programming problem (CVP) and $\overline{x} \in D$. The multiobjective programming problem (CVP) is said to satisfy the Kuhn-Tucker constraint qualification at $\overline{x}$ if,

$$C(D, \overline{x}) = \left\{d \in R^n \colon \nabla g_j(\overline{x})\, d \leqq 0,\ j \in J(\overline{x}),\ \nabla h_t(\overline{x}) d = 0,\ t \in T\right\},$$

where $C(D, \overline{x})$ represents the Bouligand tangent cone to D at $\overline{x}$.

Now, in a natural way, we extend the so-called G-Karush-Kuhn-Tucker necessary optimality conditions (see [4]) to the vectorial case, that is, for differentiable multiobjective programming problems.

Theorem 18 *(G-Karush-Kuhn-Tucker necessary optimality conditions). Let $\overline{x} \in D$ be a (weak) Pareto optimal point in problem (CVP). Moreover, we assume that G_{f_i}, $i \in I$, is a differentiable real-valued strictly increasing function defined on $I_{f_i}(D)$, G_{g_j}, $j \in J$, is a differentiable real-valued strictly increasing function defined on $I_{g_j}(D)$, and G_{h_t}, $t \in T$, is a differentiable real-valued strictly increasing function defined on $I_{h_t}(D)$ such that the Kuhn-Tucker constraint qualification is satisfied at $\overline{x}$ for (CVP). Then, there exist $\overline{\lambda} \in R^k$, $\overline{\xi} \in R^m$ and $\overline{\mu} \in R^p$ such that*

$$\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}(f_i(\overline{x}))\, \nabla f_i(\overline{x}) + \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{x})\right) \nabla g_j(\overline{x}) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}(h_t(\overline{x}))\, \nabla h_t(\overline{x}) = 0, \tag{6}$$

$$\overline{\xi}_j \left[G_{g_j}\left(g_j(x)\right) - G_{g_j}\left(g_j(\overline{x})\right)\right] \leqq 0, \quad j \in J, \quad \forall x \in D, \tag{7}$$

$$\overline{\lambda} \geq 0, \overline{\xi} \geqq 0. \tag{8}$$

Proof By assumption, $\overline{x} \in D$ is a Pareto optimal point (a weak Pareto optimal point) in problem (CVP). Let d be a Bouligand tangent vector for the set D at $\overline{x}$ and (x_l) is the corresponding sequence of feasible solutions in (CVP) converging to $\overline{x}$ and (λ_l) be the corresponding sequence of scalars such that $\lambda_l \geq 0$ for each integer l (see Definition 14). We denote by $W = \left\{\left(G_{f_1}(f_1(x)), \ldots, G_{f_k}(f_k(x))\right) : x \in D\right\} \subset R^k$ and $\overline{z} = \left(G_{f_1}(f_1(\overline{x})), \ldots, G_{f_k}(f_k(\overline{x}))\right) \subset W$. Since $\overline{x} \in D$ is a Pareto optimal point (a weak Pareto optimal point) in problem (CVP) and G_{f_i}, $i \in I$, is a differentiable real-valued strictly increasing function defined on $I_{f_i}(D)$, then $\overline{z}$ is a (weak) minimal vector in the set W (Theorem 13). Further, we consider the sequence of vectors $(z_l) \subset W$, where $z_l = \left(G_{f_1}(f_1(x_l)), \ldots, G_{f_k}(f_k(x_l))\right) \in W$. From the differentiability of f_i and G_{f_i} at $\overline{x}$, we have, for any $i \in I$,

$$G_{f_i}(f_i(x_l)) - G_{f_i}(f_i(\overline{x})) = G'_{f_i}(f_i(\overline{x}))\,\nabla f_i(\overline{x})\,(x_l - \overline{x}) + \alpha_i(\|x_l - \overline{x}\|), \tag{9}$$

where

$$\frac{\alpha_i(\|x_l - \overline{x}\|)}{\|x_l - \overline{x}\|} \to 0, \; x_l \to \overline{x}.$$

Now, we find some convergence vector for the set W. By (9), we have, for any $i \in I$,

$$\begin{aligned}\frac{z_l - \overline{z}}{\lambda_l} &= \frac{1}{\lambda_l}\left(\begin{bmatrix} G_{f_1}(f_1(x_l)) \\ \vdots \\ G_{f_k}(f_k(x_l)) \end{bmatrix} - \begin{bmatrix} G_{f_1}(f_1(\overline{x})) \\ \vdots \\ G_{f_k}(f_k(\overline{x})) \end{bmatrix}\right) \\ &= \begin{bmatrix} G'_{f_1}(f_1(\overline{x}))\,\nabla f_1(\overline{x}) \\ \cdots \\ G'_{f_k}(f_k(\overline{x}))\,\nabla f_k(\overline{x}) \end{bmatrix} \frac{(x_l - \overline{x})}{\lambda_l} + \frac{\alpha(\|x_l - \overline{x}\|)}{\|x_l - \overline{x}\|} \cdot \frac{\|x_l - \overline{x}\|}{\lambda_l}. \end{aligned} \tag{10}$$

By assumption, (x_l) is a sequence of feasible solutions in (CVP) converging to $\overline{x}$. In view of differentiability of the functions f_i and G_{f_i}, $i \in I$, at $\overline{x}$ it follows that they are also continuous functions at $\overline{x}$. Therefore, the sequence $(z_l) \in W$ converges to $\overline{z} = \left(G_{f_1}(f_1(\overline{x})), \ldots, G_{f_k}(f_k(\overline{x}))\right)$. Hence, by (10), it follows that

$$q = \lim_{l \to \infty} \frac{z_l - \overline{z}}{\lambda_l} = \begin{bmatrix} G'_{f_1}(f_1(\overline{x}))\,\nabla f_1(\overline{x}) \\ \cdots \\ G'_{f_k}(f_k(\overline{x}))\,\nabla f_k(\overline{x}) \end{bmatrix} d \tag{11}$$

Then, by Definition 14, q is a Bouligand tangent vector for the set W at $\overline{z}$.

Since the Kuhn-Tucker constraint qualification is satisfied at $\overline{x}$ and d is a Bouligand tangent vector for D at $\overline{x}$, then we have

$$\begin{aligned} \nabla g_j(\overline{x})\,d &\leqq 0, \quad j \in J(\overline{x}), \\ \nabla h_t(\overline{x}) d &= 0, \quad t \in T. \end{aligned}$$

By assumption, G_{g_j}, $j \in J$, is a differentiable real-valued strictly increasing function defined on $I_{g_j}(D)$, and G_{h_t}, $t \in T$, is a differentiable real-valued strictly increasing function defined on $I_{h_t}(D)$. Thus,

$$\begin{aligned} G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x})\,d &\leqq 0, \quad j \in J(\overline{x}), \\ G'_{h_t}(h_t(\overline{x}))\,\nabla h_t(\overline{x}) d &= 0, \quad t \in T. \end{aligned} \tag{12}$$

Since $\overline{z}$ is a (weak) minimal vector in the set W, then, by Theorem 16, there is no Bouligand tangent vector for the set W at the point $\overline{z}$ strictly negative. Thus, by (11) and (12), we obtain the following system

$$\begin{aligned} G'_{f_i}(f_i(\overline{x}))\,\nabla f_i(\overline{x}) d &< 0, \quad i \in I, \\ G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x}) d &\leqq 0, \quad j \in J(\overline{x}), \\ G'_{h_t}(h_t(\overline{x}))\,\nabla h_t(\overline{x}) d &= 0, \quad t \in T, \end{aligned}$$

which is inconsistent. From Motzkin's theorem [18], it follows that the system

$$\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i(\overline{x})\right) \nabla f_i\left(\overline{x}\right) + \sum_{j \in J(\overline{x})} \overline{\zeta}_j G'_{g_j}\left(g_j(\overline{x})\right) \nabla g_j\left(\overline{x}\right)$$
$$+ \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{x})\right) \nabla h_t(\overline{x}) = 0, \overline{\lambda} \in R^k, \overline{\lambda} \geq 0, \overline{\zeta} \in R^{J(\overline{x})}, \overline{\zeta} \geqq 0, \overline{\mu} \in R^p$$

is consistent. Let $\left(\overline{\lambda}, \overline{\zeta}, \overline{\mu}\right)$ be a solution to the system above. Then, we define $\overline{\xi} \in R^m$ as follows:

$$\overline{\xi}_j = \overline{\zeta}_j,\ j \in J\left(\overline{x}\right),$$
$$\overline{\xi}_j = 0,\ j \notin J\left(\overline{x}\right).$$

Thus, we conclude that $\left(\overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ satisfies the G-Karush-Kuhn-Tucker necessary optimality conditions (6)–(8), and so, this theorem is proved. □

Before we prove the sufficiency of the G-Karush-Kuhn-Tucker necessary optimality conditions for the considered multiobjective programming problems with functions belonging to the introduced class of nonconvex functions, we introduce the following denotations of two set of equality constraints indices. Namely, we denote by $T^+(\overline{x})$ and $T^-(\overline{x})$ the sets of equality constraints indices for which a corresponding Lagrange multiplier is positive and negative, respectively, that is, $T^+(\overline{x}) = \left\{t \in T\colon \overline{\mu}_t > 0\right\}$ and $T^-(\overline{x}) = \left\{t \in T\colon \overline{\mu}_t < 0\right\}$.

Now, we establish the sufficient optimality conditions for multiobjective programming problems of such a type. In the first theorem, we assume that the functions constituting the considered vector optimization problem (CVP) belong to the introduced class of nonconvex functions. Then we prove that a feasible point $\overline{x}$, at which the G-Karush-Kuhn-Tucker necessary optimality conditions are fulfilled, is a weak Pareto optimal point. In the next theorem, under stronger assumption imposed on the functions constituting problem (CVP), we also give the sufficient conditions for Pareto optimality.

Theorem 19 *Let $\overline{x}$ be a feasible point for (CVP), G_{f_i}, $i \in I$, be a differentiable real-valued strictly increasing function defined on $I_{f_i}(D)$, G_{g_j}, $j \in J$, be a differentiable real-valued strictly increasing function defined on $I_{g_j}(D)$, and G_{h_t}, $t \in T$, be a differentiable real-valued strictly increasing function defined on $I_{h_t}(D)$, such that the Kuhn-Tucker constraint qualification and the G-Karush-Kuhn-Tucker necessary optimality conditions (6)–(8) are satisfied at $\overline{x}$. Further, assume that f is vector G_f-invex with respect to η at $\overline{x}$ on D, g is vector G_g-invex with respect to the same function η at $\overline{x}$ on D, h_t, $t \in T^+(\overline{x})$, is G_{h_t}-invex with respect to η at $\overline{x}$ on D, and h_t, $t \in T^-(\overline{x})$, is G_{h_t}-incave with respect to η at $\overline{x}$ on D. Then $\overline{x}$ is a weak Pareto optimal point in (CVP).*

Proof Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto optimal point for (CVP). Hence, there exists $\widetilde{x} \in D$ such that

$$f\left(\widetilde{x}\right) < f\left(\overline{x}\right). \tag{13}$$

By assumption, f is G_f-invex with respect to the same function η at $\overline{x}$ on D. Then, by Definition 2, there exist functions $G_{f_i}\colon I_{f_i}(D) \to R$, $i \in I$, which are strictly increasing on their domains. Hence, (13) yields, for any $i \in I$,

$$G_{f_i}\left(f_i\left(\widetilde{x}\right)\right) < G_{f_i}\left(f_i\left(\overline{x}\right)\right). \tag{14}$$

Then, by Definition 2, we have, for any $i \in I$,

$$G_{f_i}\left(f_i(\widetilde{x})\right) - G_{f_i}\left(f_i(\overline{x})\right) \geqq G'_{f_i}\left(f_i(\overline{x})\right) \nabla f_i(\overline{x}) \eta(\widetilde{x}, \overline{x}), \tag{15}$$

By (14) and (15),

$$G'_{f_i}\left(f_i(\overline{x})\right) \nabla f_i(\overline{x}) \eta(\widetilde{x}, \overline{x}) < 0. \tag{16}$$

By assumption, $\overline{x}$ is such a feasible solution for (CVP), at which the G-Karush-Kuhn-Tucker necessary optimality conditions are satisfied. Thus, by the G-Karush-Kuhn-Tucker (8) and (16), it follows that

$$\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i(\overline{x})\right) \nabla f_i(\overline{x}) \eta(\widetilde{x}, \overline{x}) < 0. \tag{17}$$

By assumption, g is G_g-invex with respect to the same function η at $\overline{x}$ on D. Then by Definition 2, we have, for any $j \in J$,

$$G_{g_j}\left(g_j(\widetilde{x})\right) - G_{g_j}\left(g_j(\overline{x})\right) \geqq G'_{g_j}\left(g_j(\overline{x})\right) \nabla g_j(\overline{x}) \eta(\widetilde{x}, \overline{x}). \tag{18}$$

Thus, by the G-Karush-Kuhn Tucker necessary optimality condition (8), it follows that

$$\overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) - \overline{\xi}_j G_{g_j}\left(g_j(\overline{x})\right) \geqq \overline{\xi}_j G'_{g_j}\left(g_j(\overline{x})\right) \nabla g_j(\overline{x}) \eta(\widetilde{x}, \overline{x}).$$

Then, the G-Karush-Kuhn-Tucker necessary optimality condition (7) implies

$$\overline{\xi}_j G'_{g_j}\left(g_j(\overline{x})\right) \nabla g_j(\overline{x}) \eta(\widetilde{x}, \overline{x}) \leqq 0, \tag{19}$$

and so,

$$\sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{x})\right) \nabla g_j(\overline{x}) \eta(\widetilde{x}, \overline{x}) \leqq 0. \tag{20}$$

By assumption, h_t, $t \in T^+(\overline{x})$, is G_{h_t}-invex with respect to η at $\overline{x}$ on D, and h_t, $t \in T^-(\overline{x})$, is G_{h_t}-incave with respect to η at $\overline{x}$ on D. Then, by Definition 2, we have,

$$G_{h_t}\left(h(\widetilde{x})\right) - G_{h_t}\left(h(\overline{x})\right) - G'_{h_t}\left(h(\overline{x})\right) \nabla h_t(\overline{x}) \eta(\widetilde{x}, \overline{x}) \geqq 0, \quad t \in T^+(\overline{x}),$$

$$G_{h_t}\left(h(\widetilde{x})\right) - G_{h_t}\left(h(\overline{x})\right) - G'_{h_t}\left(h(\overline{x})\right) \nabla h_t(\overline{x}) \eta(\widetilde{x}, \overline{x}) \leqq 0, \quad t \in T^-(\overline{x}).$$

Thus, for any $t \in T$,

$$\overline{\mu}_t G_{h_t}\left(h(\widetilde{x})\right) - \overline{\mu}_t G_{h_t}\left(h(\overline{x})\right) - \overline{\mu}_t G'_{h_t}\left(h(\overline{x})\right) \nabla h_t(\overline{x}) \eta(\widetilde{x}, \overline{x}) \geqq 0$$

Since $\widetilde{x} \in D$ and $\overline{x} \in D$, then the inequality above implies

$$\sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h(\overline{x})\right) \nabla h_t(\overline{x}) \eta(\widetilde{x}, \overline{x}) \leqq 0. \tag{21}$$

Adding both sides of inequalities (17), (20) and (21), we get the inequality

$$\Big[\textstyle\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i(\overline{x})\right) \nabla f_i(\overline{x}) + \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{x})\right) \nabla g_j(\overline{x}) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h(\overline{x})\right) \nabla h_t(\overline{x})\Big] \eta(\widetilde{x}, \overline{x}) < 0,$$

which contradicts the G-Karush-Kuhn-Tucker necessary optimality condition (6). Hence, $\overline{x}$ is a weak Pareto optimal for (CVP), and the proof is complete. □

Theorem 20 *Let $\overline{x}$ be a feasible point for (CVP). Assume that there exist $G_{f_i}, i \in I$, is a differentiable real-valued strictly increasing function defined on $I_{f_i}(D)$, $G_{g_j}, j \in J$, is a differentiable real-valued strictly increasing function defined on $I_{g_j}(D)$, and $G_{h_t}, t \in T$, is a differentiable real-valued strictly increasing function defined on $I_{h_t}(D)$, such that the Kuhn-Tucker constraint qualification and the G-Karush-Kuhn-Tucker necessary optimality conditions (6)–(8) are satisfied at $\overline{x}$. If f is vector strictly G_f-invex with respect to η at $\overline{x}$ on D, g is vector G_g-invex with respect to η at $\overline{x}$ on D, $h_t, t \in T^+(\overline{x})$, is G_{h_t}-invex with respect to η at $\overline{x}$ on D, and $h_t, t \in T^-(\overline{x})$, is G_{h_t}-incave with respect to η at $\overline{x}$ on D, then $\overline{x}$ is a Pareto optimal point in (CVP).*

Proof Proof for Pareto optimality is similar as the proof of Theorem 19. □

Remark 21 Note that to prove Theorem 19 it is sufficient to assume that $f_i, i \in I(\overline{x})$, are G_{f_i}-invex with respect to the same function η at $\overline{x}$ on D, $g_{J(\overline{x})}, j \in J(\overline{x})$, are G_{g_j}-invex with respect to the same function η at $\overline{x}$ on D in place of f is vector G_f-invex with respect to η at $\overline{x}$ on D and g is vector G_g-invex with respect to η at $\overline{x}$ on D. Also to prove Theorem 20 it is sufficient to assume that $f_i, i \in I(\overline{x})$, are G_{f_i}, invex with respect to the same function η at $\overline{x}$ on D, at least one $f_i, i \in I(\overline{x})$, is strictly G_{f_i}, invex with respect to the same function η at $\overline{x}$ on D, and $g_{J(\overline{x})}, j \in J(\overline{x})$, are G_{g_j}-invex with respect to the same function η at $\overline{x}$ on D in place of f is strictly vector G_f-invex with respect to η at $\overline{x}$ on D and g is vector G_g-invex with respect to η at $\overline{x}$ on D.

Remark 22 Note that if the Lagrange multiplier $\overline{\lambda}$ associated with the objective function in the considered multiobjective programming problem (CVP) is assumed to satisfy $\overline{\lambda} > 0$ then, to prove Theorem 20, it is also sufficient to assume, in place of strictly G_f-invexity of f, that the objective function f is vector G_f-invex with respect to the same function η as other functions constituting problem (CVP).

Now, we illustrate the optimality results established in the paper by a suitable multiobjective programming problem involving vector G-invex functions with respect to the same function η, but with respect to not the same function G.

Example 1 We now consider the following multiobjective programming problem

$$\min \quad f(x) = \left(e^{x^2-4x}, \arctan x\right)$$
$$g(x) = \ln\left(x^2 - x + 1\right) \leqq 0.$$

For the above multiobjective programming problem, we have $D = [0, 1]$. Note that $\overline{x}^1 = 0$ and $\overline{x}^2 = 1$ are Pareto optimal solutions in the considered multiobjective programming problem. It is not difficult to prove, by Definition 2, that f is vector G_f-invex function with respect to η and g is G_g-invex function with respect to η at $\overline{x}^1 = 0$ and $\overline{x}^2 = 1$ on D, where, for example, η is defined by

$$\eta(x, \overline{x}) = -\frac{1}{4}x^2 + x - \overline{x}\left(\frac{1}{2}\overline{x} + 1 - x + \frac{1}{4}x^2\right).$$

and, moreover,

$$G_{f_1}(t) = \frac{1}{2}\ln t, \ G_{f_2}(t) = \tan t, \tag{22}$$

$$G_g(t) = e^t. \tag{23}$$

It is not difficult to verify that the G-Karush-Kuhn-Tucker necessary optimality conditions (6)–(8) with the functions G_f and G_g and the Kuhn-Tucker constraint qualification are satisfied at the feasible points $\overline{x}^1 = 0$ and $\overline{x}^2 = 1$. Since all hypotheses of Theorem 20 are fulfilled, then we can use the sufficient optimality conditions from this theorem to show that $\overline{x}^1 = 0$ and $\overline{x}^2 = 1$ are Pareto optimal solutions in the considered multiobjective programming problem (CVP). Further, note that the sufficient optimality conditions for Pareto optimality valid for convex vector optimization problems (see, for example, [22]) are not applicable for the considered multiobjective programming problem (CVP). This follows from the fact that not all functions involving in the considered multiobjective programming problem are convex. Also note that the functions constituting the considered multiobjective programming problem are not invex at $\overline{x}^1 = 0$ and $\overline{x}^2 = 1$ on D with respect to the function η defined above. Therefore, also the sufficient optimality conditions applicable for invex vector optimization problems are not valid in this case.

As follows from this example, in some cases of differentiable nonconvex vector optimization problems, it is in an easier way to find such a vector-valued function η with respect to which all functions are vector G-invex on D than vector invex on D. Therefore, in some cases, to prove (weak) Pareto optimality in some classes of differentiable vector optimization problems, vector G-invexity is more useful than vector invexity notion .

4 Conclusion

This paper represents the first part of a study concerning the so-called G-multiobjective programming. We have proved new necessary and sufficient optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints. It is pointed out that our statement of the so-called G-Karush-Kuhn-Tucker necessary optimality conditions established in this work is more general than the classical Karush-Kuhn-Tucker necessary optimality conditions found in the literature. Furthermore, we have proved the sufficiency of the introduced G-Karush-Kuhn-Tucker necessary optimality conditions for nonconvex multiobjective programming problems. More exactly, this result has been proved for such multiobjective programming problems in which the objective functions, the inequality constraints and the equality constraints (for which associated Lagrange multipliers are positive) are G-invex with respect to the same function η and the equality constraints (for which associated Lagrange multipliers are negative) are G-incave with respect to the same function η, but not necessarily with respect to the same function G. We have illustrated the results proved in the paper by the suitable example of a multiobjective programming problem involving functions of this type. Moreover, by the help of this example, we have illustrated also the fact that in some cases, to prove (weak) Pareto optimality in some classes of differentiable vector optimization problems, vector G-invexity is more useful than other vector generalized convexity notions.

References

1. Antczak, T.: (p, r)-invex sets and functions. J. Math. Anal. Applic. **263**, 355–379 (2001)
2. Antczak, T.: (p, r)-invexity in multiobjective programming. Eur. J. Oper. Res. **152**, 72–87 (2004)
3. Antczak, T.: The notion of V-r-invexity in differentiable multiobjective programming. J. Appl. Anal. **11**, 63–79 (2005)
4. Antczak, T.: New optimality conditions and duality results of G-type in differentiable mathematical programming. Nonlinear Anal. **66**, 1617–1632 (2007)

5. Batista dos Santos, L., Osuna-Gomez, R., Rojas-Medar, M.A., Rufian-Lizana, A.: Invexity generalized and weakly efficient solutions for some vectorial optimization problem in Banach spaces. Numer. Funct. Anal. Optim. (2004)
6. Bazaraa, M.S., Sherali, H.D., Shetty, C.M.: Nonlinear programming: theory and algorithms. Wiley, New York (1991)
7. Ben-Israel, A., Mond, B.: What is invexity? J. Aust. Math. Soc. Ser. B **28**, 1–9 (1986)
8. Craven, B.D.: Invex functions and constrained local minima. Bulletin of the Aust. Math. Soc. **24**, 357–366 (1981)
9. Craven, B.D., Glover, B.M.: Invex functions and duality. J. Aust. Math. Soc. Ser .A **39**, 1–20 (1985)
10. Geoffrion, M.A.: Proper efficiency and the theory of vector maximization. J. Math. Anal. Applic. **22**, 613–630 (1968)
11. Giorgi, G., Guerraggio, A.: The notion of invexity in vector optimization: smooth and nonsmooth case. In: Crouzeix, J.P. et al. (eds.), Generalized Convexity, Generalized Monotonicity. Kluwer Academic Publishers (1998)
12. Hanson, M.A.: On sufficiency of the Kuhn-Tucker conditions. J. Math. Anal. Applic. **80**, 545–550 (1981)
13. Hanson, M.A., Mond, B.: Necessary and sufficient conditions in constrained optimization. Math. Programming **37**, 51–58 (1987)
14. Jeyakumar, V., Mond, B.: On generalized convex mathematical programming. J. Aust. Mathe. Soc. Ser. B **34**, 43–53 (1992)
15. Kanniappan, P.: Necessary conditions for optimality of nondifferentiable convex multiobjective programming. J. Optim. Theory Applic. **40**, 167–174 (1983)
16. Kaul, R.N., Suneja, S.K., Srivastava, M.K.: Optimality criteria and duality in multiple objective optimization involving generalized invexity. J. Optim. Theory Applic. **80**, 465–482 (1994)
17. Lin, J.G.: Maximal vectors and multi-objective optimization. J. Optim. Theory Applic. **18**, 41–64 (1976)
18. Mangasarian, O.L.: Nonlinear programming. McGraw-Hill, New York (1969)
19. Martin, D.H.: The essence of invexity. J. Optim. Theory Applic. **47**, 65–76 (1985)
20. Osuna-Gómez, R., Rufián-Lizana, A., Ruiz-Canales, P.: Invex functions and generalized convexity in multiobjective programming. J. Optim. Theory Applic. **98**, 651–661 (1998)
21. Pareto, V.: Cours de economie politique. Rouge, Lausanne, Switzerland (1896)
22. Singh, C.: Optimality conditions in multiobjective differentiable programming. J. Optim. Theory Applic. **53**, 115–123 (1987)
23. Varaiya, P.P.: Notes on optimization. Van Nostrand Reinhold Company, New York, New York (1972)
24. Weir, T.: Proper efficiency and duality for vector valued optimization problems. J. Aust. Math. Soc. Ser. A **43**, 24–34 (1987)